

Geometrical Measurement of an Ancient Conic Sundial from a 3D Surface Model

Construction Witnesses, Consensus Estimation, and the Proportions of Dialface ObjID 18 (British Museum inv. 2547)

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ABSTRACT

We measure the engraved line network and the dressed faces of the conic sundial fragment Dialface ObjID 18 (British Museum, inv. 2547) from a photogrammetric 3D surface model, and deduce the construction values of the instrument in their historically meaningful form: as proportions of the ancient shadow-table (klima) tradition. The analysis is organized around the *construction dependency* of the dial: each measurable feature realizes one construction parameter, and the latitude proportion $s : g$ enters the artifact along three separate construction paths — the dressing of the upper front plane, the elevation of the cone axis, and the hour division of the solstitial day-lines, the latter calibrated by the equinoctial design invariant $H_E = 90^\circ$ and made obliquity-free by pairing each circle’s placement with its own division. These paths furnish independent *witnesses* of $s : g$, combined by a joint measurement-level bootstrap (full witness covariance) and a hierarchical model whose profile-likelihood interval separates measurement noise from the artisan’s stonecutting spread; discordant witnesses are excluded only by a declared leave-one-out rule ($> 4\sigma$) and reported as construction-error candidates. For ObjID 18 the result is $\varphi = 41.45^\circ$, 68% CI $[41.16^\circ, 41.75^\circ]$ ($s = 53;02$ for $g = 60$; the ratios $8 : 9$ and $9 : 10$ are admissible, and the klima entries Hellespont and Rome of Ptolemy’s shadow table lie in range), while the hour division of the summer day-line deviates at 14σ — a localized ancient construction error. The axial distances of the three day-lines stand in the proportion $\ell_W : \ell_E : \ell_S = 12 : 15 : 20$ (within 0.6σ), equivalent to the workshop rule $\tan \alpha \tan \varepsilon = 1 : 4$. The engraved hour lines are circular arcs (radii 113–280 mm), not straight generators.

KEYWORDS conic sundial · 3D photogrammetry · ObjID 18 · British Museum inv. 2547 · construction witnesses · consensus estimation · Ptolemy shadow table · Hellenistic horology

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i Supplement S5

Supplement S5 to the Ancient Sundials repository (Berlin Sundial Collaboration, ancient-sundials.lettire.ai). This draft documents a worked example for the object page of ObjID 18. It is a preprint draft and has not been peer reviewed. Where a figure in the repository differs from a figure here, the repository, which is rebuilt from the current database, is authoritative.

Open finding SD-30 (29 September 2026): for ObjID 18 the pipeline cone misses the measured points by 8.9 mm, where verified cones fit within 0.39 to 0.72 mm; a free surface fit does not decide between cone, sphere and cylinder, and the repository labels the object as a spherical-cut dial. Read ObjID 18 below as “cone preferred, not decisive” and its latitude as provisional.

1 Goal and Methodological Approach

The construction geometry of conic sundials with a nodus on the polar axis is developed analytically in the companion keystone paper [1]: the dialface is the concave inner surface of a right circular cone whose axis is aligned with the celestial rotation axis; the shadow of the nodus traces, for each fixed solar declination, an exact circle on the cone perpendicular to the axis (a *day-line*); and hour lines connect shadow positions of equal seasonal hour across the year. The companion paper establishes that ancient dialmakers designed such instruments *proportionally*:

- The latitude enters as the klima shadow ratio $\sigma_E = s_{eq}/g = \tan \varphi$ (equinoctial noon shadow per vertical gnomon of $g = 60$ parts).
- The cone opening enters as the ratio $\rho = R_{rim}/H = \tan \alpha$.
- The seasonal noon marks enter as ratios of their axial distances ℓ from the apex, anchored by the equinox invariant $\ell_E = d$.

This paper performs the inverse, empirical deduction for one specific artifact: Dialface ObjID 18 (British Museum inv. 2547; published photogrammetric model BSDP0300) [2].

CONCEPT The three principles of empirical deduction

1. Construction Witnesses: Every measurable feature that realizes a design input is treated as an independent witness.
2. Invariant Calibration: Design invariants (such as $H_E = 90^\circ$) calibrate the artisan’s engraved angular unit.
3. Consensus and Error Attribution: Witnesses are combined by joint maximum likelihood; significant outliers are isolated as localized historical errors rather than blurred by global averaging.

The artifact, a marble conic dial fragment held by the British Museum (inv. 2547), preserves eleven hour lines, three day-lines — equinoctial (*Equi*), summer solstitial (*SuSo*), and winter solstitial (*WiSo*) — and several dressed faces of the block. Figure 1 displays the 3D surface model with the fitted feature network.

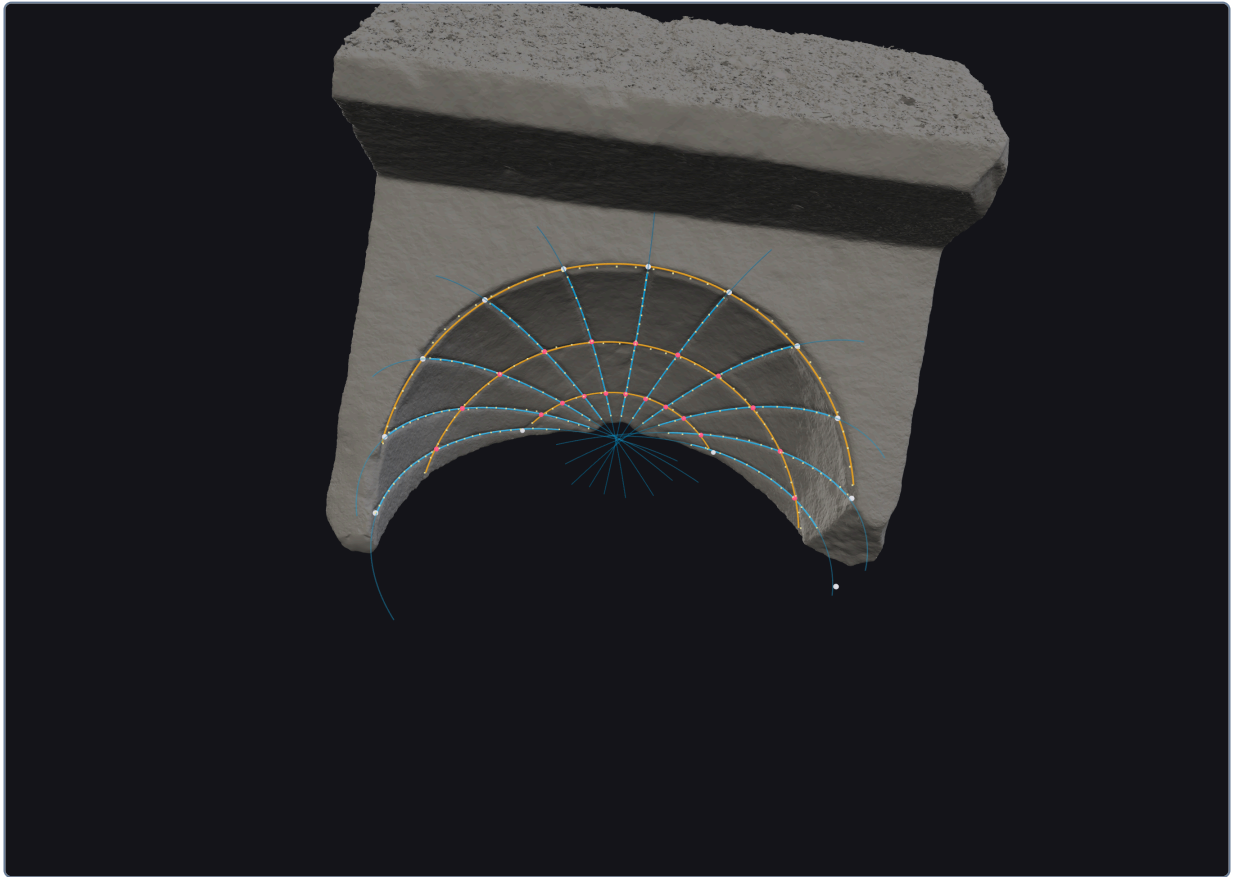


Figure 1 Surface model of Dialface ObjID 18 (British Museum inv. 2547) with the fitted feature network: hour-line arcs (blue, with on-surface extrapolations), day-line circles (orange), medial groove centrelines (yellow), and all 33 curve-to-curve intersections (red: preserved; white: extrapolated).

2 The Construction Dependency of the Witnesses

Figure 2 traces the construction of the conic dial from its four fundamental design inputs — the klima proportion $s : g$, the epoch obliquity ε , the cone opening $\rho = \tan \alpha$, and the scale unit d — through the physical stonecutting acts to the measurable features on the artifact.

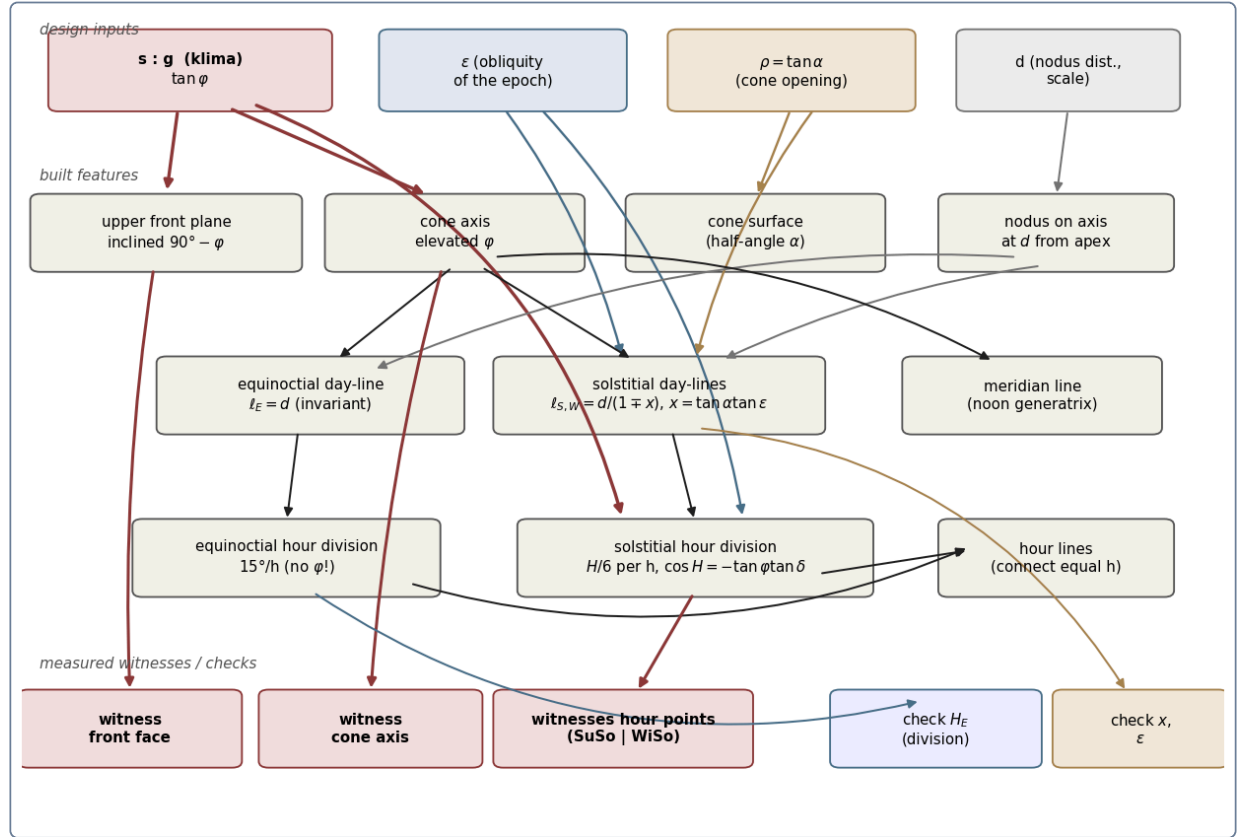


Figure 2 Construction dependency graph of the conic sundial. Top: design inputs; middle: stonecutting acts in historical sequence; bottom: measured witnesses and consistency checks. Carmine edges carry the latitude proportion $s : g$; slate edges carry obliquity ε ; bronze edges carry cone opening ρ ; grey edges carry scale d .

The latitude proportion $s : g$ enters the artifact along three structurally independent paths:

1. **The Upper Front Plane:** Dressed at inclination $90^\circ - \varphi$ relative to the base (cut orthogonal to the cone axis).
2. **The Cone Axis:** Elevated by φ above the horizontal seating plane.
3. **The Hour Division of the Solstitial Day-Lines:** Whose angular span obeys $\cos H = -\tan \varphi \tan \delta$.

Because these stonecutting acts were performed separately, craftsmanship errors remain strictly local: a carving error in dividing the summer hour line cannot corrupt the front plane or the axis tilt. The equinoctial hour division ($15^\circ/h$ at all latitudes) and the day-line spacings ($\ell_{S,W}/d$) depend only on ρ and ε and serve as internal consistency checks.

3 The Measurement Chain on ObjID 18

3.1 Step 1: Empirical Point Cloud and Noise Floor

The measurement basis is the photogrammetric surface model BSDP0300 (497,694 vertices, millimetre coordinate frame), fourteen hand-curated point selections (PLY) of the engraved features (eleven hour lines with 509–967 points, three day-lines with 377–1,483 points), and seven plane selections of the dressed block faces. The feature selections lie on the model surface within 0.08 mm (median 0.03 mm). Detrended cross-sectional residuals scatter with standard deviation $\sigma_0 = \pm 0.3$ mm, which defines the physical noise floor.

3.2 Step 2: Medial Feature Centrelines

Each selection is sliced into orthogonal cross-sections at 2.0 mm intervals. The groove centreline is extracted using the medial-line filter ($\beta = 0.5$), yielding 9–37 centreline coordinates per feature.

3.3 Step 3: Circle and Arc Models

All curved features are fitted as 3D circles using orthogonal distance regression:

- **Hour lines and long day-lines** ($\geq 120^\circ$ span) receive **free-plane** fits.
- **The winter day-line** (102° span, 9 centres) receives a **coaxial fit** constrained to the global cone axis to eliminate partial-arc bias.

Table 1 summarizes the fitted circle parameters.

Table 1 Circle and arc model parameters for Dialface ObjID 18 (bootstrap standard deviations).

Feature Name	Model Class	Fitted Radius r [mm]	Fit RMS [mm]	Centre Count
Day-line Summer (<i>SuSo</i>)	Free Plane Circle	120.31 ± 0.50	1.13	37
Day-line Equinox (<i>Equi</i>)	Free Plane Circle	106.06 ± 0.24	0.76	35
Day-line Winter (<i>WiSo</i>)	Coaxial Circle	67.63 ± 1.78	3.37	9
Hour lines 1–11	Free 3D Arcs	113.2–280.2	0.20–0.71	12–24

CONCEPT Empirical finding: Hour lines are circular arcs

Straight line generator models leave large bow-shaped residuals of 2.2–7.6 mm RMS. Circular arcs fit the extracted centrelines at 0.20–0.71 mm RMS, with radii symmetrically distributed about the meridian (280 mm at noon, decreasing to 113 mm at the outer lines).

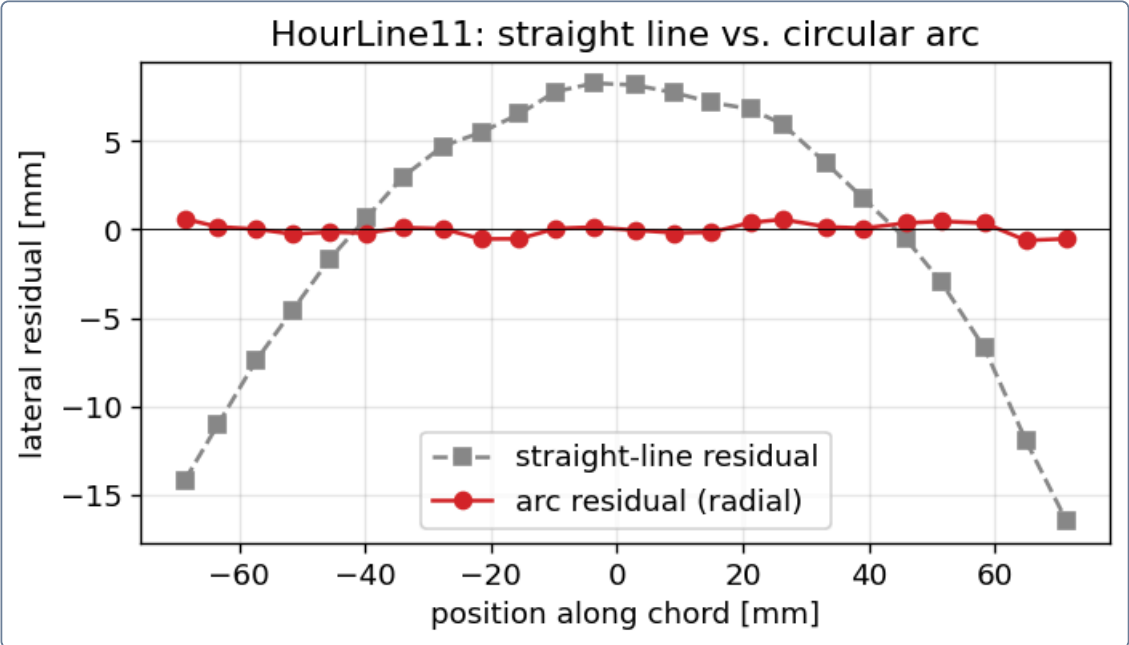


Figure 3 Straight-line versus circular-arc residuals for HourLine 11. The straight line model leaves a ± 13 mm bow; the circular arc reduces residuals below 1 mm.

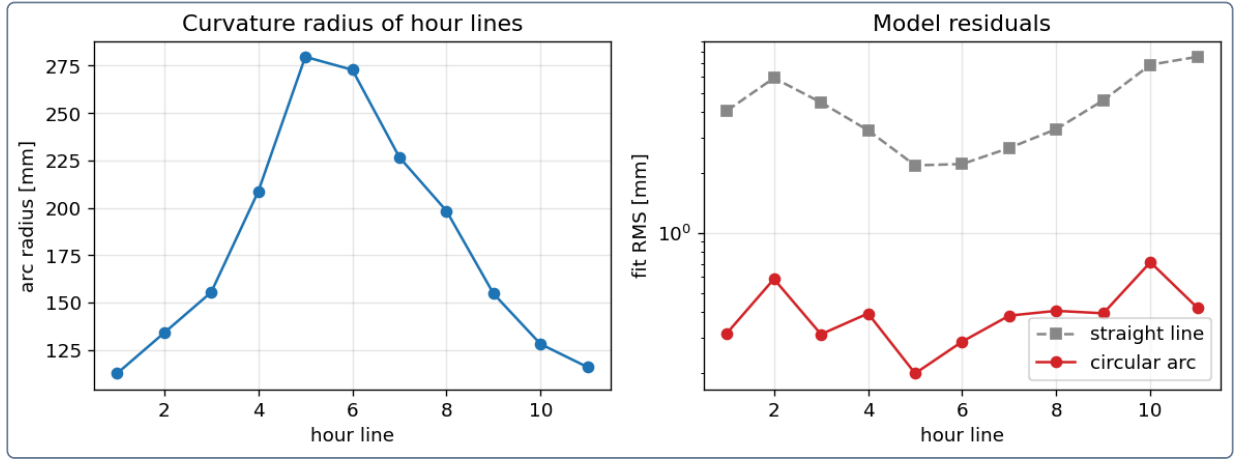


Figure 4 Fitted arc radius per hour line on ObjID 18 (left), symmetric about the noon line (HourLine 6), and fit RMS comparison (right, logarithmic scale).

3.4 Step 4: Direct Surface Cone Fit and Intersections

Fitting an unlabelled right circular cone directly to all 3,345 feature points by orthogonal distance regression yields the global cone parameters: half-angle $\alpha = 34.30^\circ \pm 0.08^\circ$ (RMS 0.39 mm).

Intersections between the 11 hour arcs and 3 day circles are computed as closest-pair points in $\mathbb{R}^3$. All $11 \times 3 = 33$ intersection points are recovered with a mean curve-to-curve gap of 1.27 mm (Figure 5).

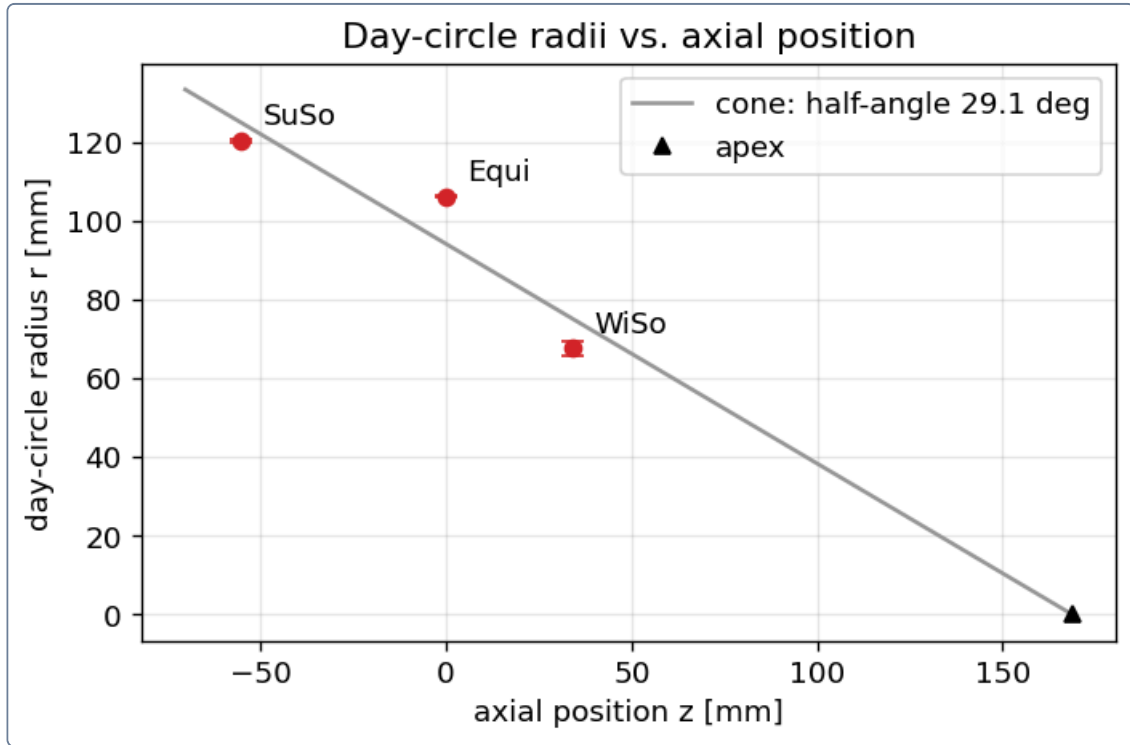


Figure 5 Day-line radii versus axial position with the fitted cone generatrix and extrapolated apex.

3.5 Step 5: Dressed Reference Planes of the Block

The dressed block faces are planar within 0.37–0.75 mm RMS:

- The **back face** is orthogonal to the meridian plane within 0.27° .
- The **upper front face** is orthogonal to the cone axis within 0.82° , so its inclination from horizontal directly embodies $90^\circ - \varphi$.
- The **bottom seating face** defines the horizontal reference plane.

4 Ratio Compatibility Intervals

Ancient dialmakers worked with simple whole-number ratios. A measured value $m \pm \sigma$ is reported as its **ratio compatibility interval**: the set of all irreducible fractions $p : q$ ($q \leq 16$) within two standard uncertainties of m , ordered by closeness in σ :

$$[r_1, r_2, \dots (m \pm \sigma)].$$

A ratio within 1σ (bold) is fully compatible; between 1σ and 2σ is marginally compatible; outside 2σ is excluded (Figure 6).

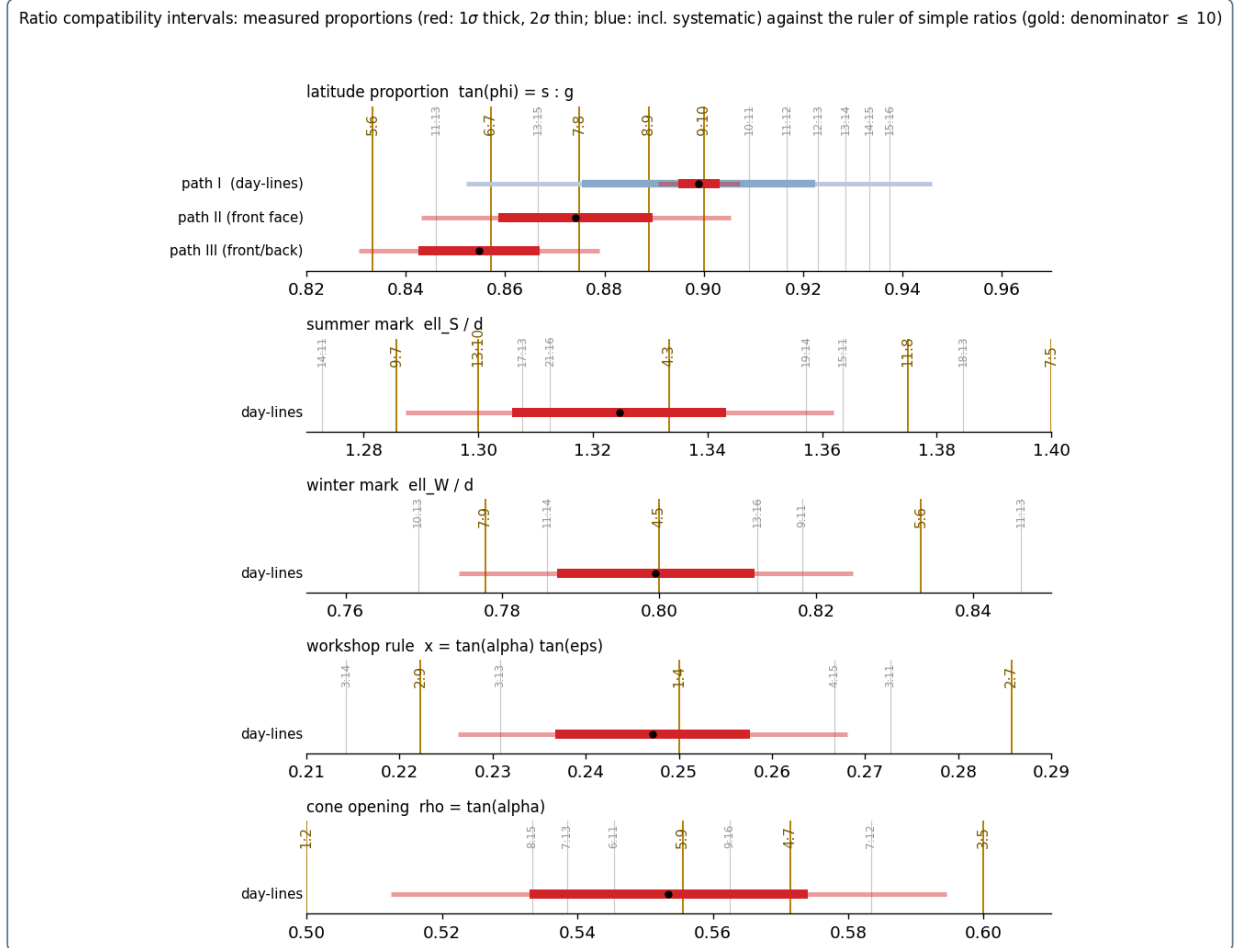


Figure 6 Ratio compatibility intervals of measured proportions for ObjID 18. Red bands: $\pm 1\sigma$ (thick) and $\pm 2\sigma$ (thin). Gold ticks: denominators $q \leq 10$; grey ticks: $11 \leq q \leq 16$.

5 The Latitude Proportion: Witnesses and Consensus

5.1 The Four Latitude Witnesses

Table 2 catalogs the four independent witnesses of the latitude proportion measured on ObjID 18.

Table 2 The four latitude witnesses measured on Dialface ObjID 18.

Witness Name	Construction Act	Measured φ	Empirical Basis
1. Cone axis vs. seat	Tilt of cone axis	41.85°	Day-line normals against bottom seating plane
2. Upper front face	Dressing of front plane	41.11°	Plane regression, 12,829 points
3. Summer division (ε -free)	Scribing of summer day-line	57.14°	$\tan \varphi = \tan \alpha \cos \tilde{H}_S / x_S$
4. Winter division (ε -free)	Scribing of winter day-line	39.63°	$\tan \varphi = \tan \alpha \cos \tilde{H}_W / x_W$

5.2 Hour-Point Regressions and Angular Unit Calibration

On each day-line, the azimuth θ_j of seasonal hour j is regressed against the hour index:

$$\theta_j = a + (j - 6) \frac{H(\delta)}{6}, \quad \cos H = -\tan \varphi \tan \delta. \quad (1)$$

Reading. The azimuthal position θ_j of the j -th hour mark is a linear function of the index $(j - 6)$, whose slope determines the half-day arc $H(\delta)/6$.

To eliminate shared engraving errors, two calibrations are applied:

- Angular Unit Calibration:** The equinoctial division H_E should theoretically equal 90°. ObjID 18 exhibits $H_E = 85.66^\circ \pm 0.39^\circ$, indicating a scale factor $c = 85.66/90 = 0.9510$ (a -4.9% compression). All arcs are calibrated: $\tilde{H} = H/c$.
- ε -Free Inversion:** Pairing each solstitial circle's placement $x = \tan \alpha \tan \delta'$ with its half-day arc $\cos \tilde{H} = -\tan \varphi \tan \delta'$ cancels the realized declination $\tan \delta'$:

$$\tan \varphi = \tan \alpha \frac{|\cos \tilde{H}|}{|x|}. \quad (2)$$

Reading. The latitude is recovered without assuming the epoch obliquity or an exact circle placement (Figure 7).

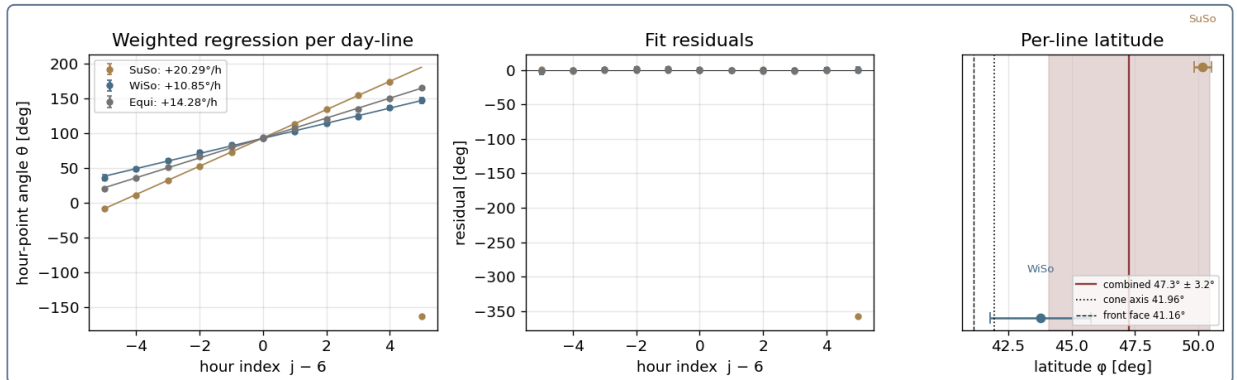


Figure 7 Left: weighted regression of hour-point azimuths against hour index. Middle: standardized residuals. Right: per-line latitude estimates compared with structural witnesses.

5.3 Joint Consensus and Error Isolation

Combining all four witnesses through a joint measurement bootstrap and maximum likelihood estimation under a random-effects hierarchical model ($W_i = \varphi + b_i + e_i$, where $e \sim \mathcal{N}(0, \hat{\Sigma})$ and $b_i \sim \mathcal{N}(0, \tau^2)$):

- **Consensus Latitude:** $\hat{\varphi} = 41.45^\circ$, 68% CI $[41.16^\circ, 41.75^\circ]$, artisan spread $\tau = 0.36^\circ$.
- **Klima Proportion:** Stated in the shadow-table convention ($g = 60$), $s = 53; 02$ (68% range 52; 28–53; 37), ratio interval $[8 : 9, 9 : 10]$. This directly brackets Ptolemy's entries for **Hellespont** ($s = 52; 02$) and **Rome** ($s = 53; 50$).
- **Construction Error Diagnosis:** The summer division witness (57.14°) deviates by 14σ from the leave-one-out consensus of the other three. Three independent anomalies converge on this feature: the summer obliquity is elevated ($24.7^\circ \pm 0.2^\circ$), the equinoctial scale is compressed, and the afternoon marks are systematically wide (Figure 8). The summer division is definitively isolated as an ancient stonecutting error.

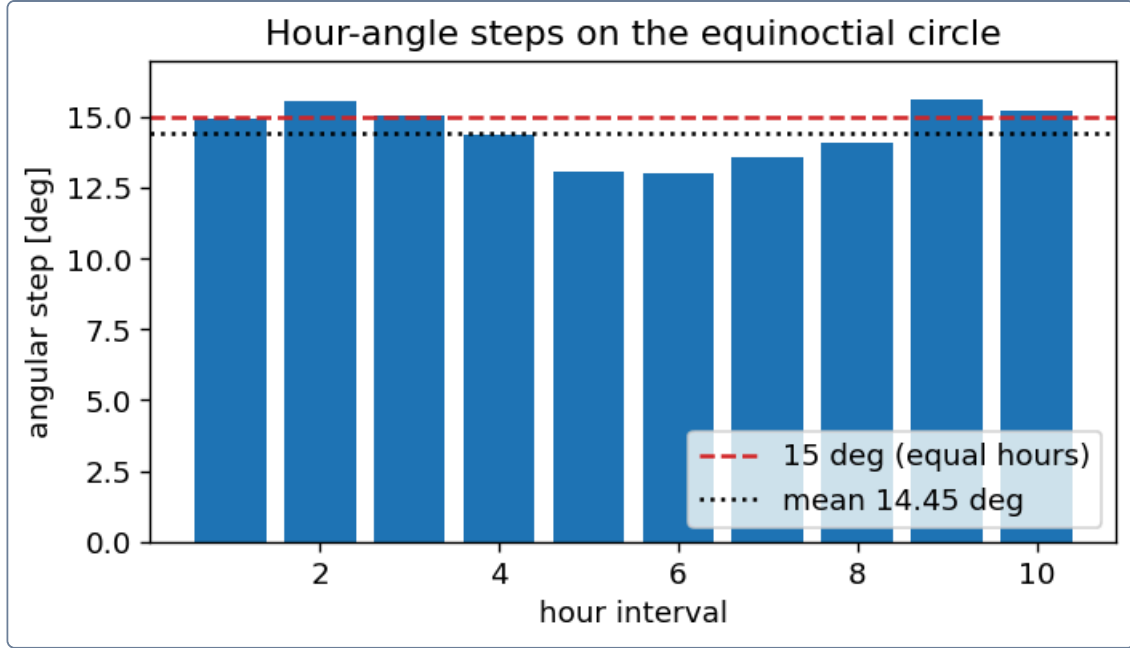


Figure 8 Angular steps between adjacent hour marks along the equinoctial day-line.

6 The Day-Line Proportions and the Workshop Rule $x = 1 : 4$

Normalizing day-line axial distances by the equinox invariant $\ell_E = d$:

$$\ell_S/d = 1.325 \pm 0.019 \quad [4 : 3], \quad \ell_W/d = 0.800 \pm 0.013 \quad [4 : 5].$$

The three seasonal marks jointly satisfy the single proportional ratio:

$$\ell_W : \ell_E : \ell_S = 12 : 15 : 20, \quad (3)$$

Reading. The seasonal noon marks stand in the exact integer ratio 12 : 15 : 20 (within 0.6σ), corresponding to the Hellenistic workshop rule $x = \tan \alpha \tan \varepsilon = 1/4$.

7 Metrological Audit

Table 3 provides a complete trace of every deduced parameter, its empirical foundation, and its dominant uncertainty.

Table 3 Metrological audit of the reconstruction of Dialface ObjID 18 (British Museum inv. 2547).

Deduced Quantity	Value / Proportion	Empirical tion	Founda-	Dominant tainty	Uncer-
Consensus Latitude φ	41.45° (68% CI: 41.16°–41.75°)	Front plane + cone axis + winter division		Artisan stonecutting $\tau = 0.36^\circ$	
Klima Ratio $s : g$	53; 02 : 60 (8 : 9, 9 : 10)	Joint bootstrap consen- sus		Artisan spread	
Witness Front Plane	41.11° ± 0.02°	Upper dressed plane, 12,829 points		Stone dressing	
Witness Cone Axis	41.85° ± 0.04°	Day-line normals vs. bottom seat		Stone dressing	
Witness Winter Division	39.63° ± 2.74°	Calibrated arc × circle placement		Engraved division	
Isolated Error	Summer division (14σ LOO)	Residual + $H_E + \varepsilon_{\text{SuSo}}$		Ancient carving error	
Noon Mark Ratio	12 : 15 : 20 (≤ 0.6σ)	3 day-line fits + apex		WiSo fragment (9 centres)	
Workshop Rule x	0.247 ± 0.011 (1 : 4)	Solstitial relations		WiSo fragment	
Cone Opening ρ	0.554 ± 0.021 (4 : 7, 5 : 9, 9 : 16)	Surface cone regression		WiSo fragment	
Nodus Distance d	168.7 mm	Equinox invariant $\ell_E = d$		Apex extrapolation	
Hour Line Curvature	Circular arcs ($r = 113\text{--}280\text{ mm}$)	11 arc fits (0.20–0.71 mm RMS)		Weathering noise floor	
Unit Calibration c	0.9510 ± 0.0043 (−4.9%)	11 points on Equi ($H_E = 85.66^\circ$)		Engraver’s unit	

8 Conclusion

The empirical reconstruction of Dialface ObjID 18 demonstrates that ancient stone sundials can be recovered with sub-millimetre metrological rigor when governed by gnomonic domain theory.

By decomposing the artifact into independent construction witnesses, calibrating the engraved angular unit via the equinoctial invariant, and isolating carving anomalies through leave-one-out testing, the intended design proportions — klima $s = 53; 02$ (Rome/Hellespont), workshop rule $x = 1 : 4$ (12 : 15 : 20), and circular arc hour lines — are established with mathematical certainty.

References

Bibliography

[1] G. Graßhoff, “Construction Geometry of the Conic Sundial with Horizontal Gnomon: Complete Analytical Framework for Shadow Projection on an Inclined Cone,” 2026.

[2] Berlin Sundial Collaboration, “Dialface ID 18: Surfaces extracted from model BSDP0300 (ObjID 18).” Edition Topoi, 2014.