

Automated Model Optimization for Sundial Feature Detectors

Nonlinear Parameter Estimation for Conic Sundial Reconstruction from 3D Scan Data

Gerd Graßhoff 

Humboldt-Universität zu Berlin, Deutsches Museum
gerd.grasshoff@hu-berlin.de

ABSTRACT

Recovering the geometric construction parameters of an ancient conic sundial from photogrammetric 3D surface scans is fundamentally a nonlinear parameter estimation problem. The raw observations are tens of thousands of point coordinates extracted from engraved chisel grooves; the mathematical model is a right circular cone constrained by gnomonic day-lines and hour networks; and the unknown structural parameters (apex location, axis orientation, cone half-angle, and nodus distance) must be recovered by minimizing an objective loss function over non-Euclidean manifolds. Grounded in the classical estimation framework of Bard (1974), this paper presents a three-layer optimization architecture: (1) observational mesh space, (2) feature-level primitive regression, and (3) global cone joint estimation. We evaluate gradient-free Nelder–Mead simplex, Trust Region Reflective (TRF), and L-BFGS-B algorithms on Dialface ObjID 18 (British Museum inv. 2547). We demonstrate how the degenerate partial-arc bias of fragmentary day-lines is resolved by direct unlabelled orthogonal distance regression over the full stone cavity, yielding stable parameter covariances and rigorous confidence intervals.

KEYWORDS nonlinear parameter estimation · conic sundials · 3D reconstruction · model optimization · Trust Region Reflective · Jacobian covariance · computational archaeology

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i Supplement S4

Supplement S4 to the Ancient Sundials repository (Berlin Sundial Collaboration, ancient-sundials.lettire.ai). This draft documents the cone fit and the latitude estimates of the measurement panel. It is a preprint draft and has not been peer reviewed. Where a figure in the repository differs from a figure here, the repository, which is rebuilt from the current database, is authoritative.

Open finding SD-30 (29 September 2026): for ObjID 18 the pipeline cone misses the measured points by 8.9 mm, where verified cones fit within 0.39 to 0.72 mm; a free surface fit does not decide between cone, sphere and cylinder, and the repository labels the object as a spherical-cut dial. Read ObjID 18 below as “cone preferred, not decisive” and its latitude as provisional.

1 Introduction: Reconstruction as Nonlinear Estimation

When an ancient conic sundial is excavated, the surviving stone is rarely intact. The apex is chipped, the gnomon is missing, and the engraved network is weathered [1], [2]. The computational task is to infer the original design parameter vector θ from noisy 3D surface measurements $\mathbf{y}_i \in \mathbb{R}^3$.

This task belongs to the classical theory of *nonlinear parameter estimation* developed by Bard [3]:

CONCEPT The estimation formulation

Let $\mathbf{y}_i \in \mathbb{R}^3$ be N observed surface coordinates along the dialface, and let $\mathbf{f}_{i(\theta)}$ be the predicted geometric position under the parameter vector $\theta = (\mathbf{A}, \hat{\mathbf{C}}, \alpha, d)$. The optimal reconstruction minimizes the weighted sum of squared residuals: $S(\theta) = \sum_{i=1}^N w_i \|\mathbf{y}_i - \mathbf{f}_{i(\theta)}\|^2$.

Naive optimization encounters three acute mathematical difficulties:

1. **Nonlinear Constraints on Rotational Manifolds:** The cone axis $\hat{\mathbf{C}}$ lies on the unit sphere S^2 , and the half-angle is bounded by the physical horizon.
2. **Partial-Arc Degeneracy:** On broken fragments where a day-line is preserved only over a short arc ($\Delta\theta < 120^\circ$), unconstrained circle fitting suffers from severe normal instability: small transverse noise pitches the fitted plane by tens of degrees.
3. **Correlated Witness Covariance:** Different physical features (the upper front plane, the axis tilt, the solstitial hour divisions) share the same cone geometry and must be estimated through a joint hierarchical model rather than isolated univariate fits.

This paper develops a robust three-layer optimization framework that resolves these challenges and yields rigorous parameter confidence intervals.

2 The Three-Layer Estimation Architecture

Following Bard’s formulation, we structure the parameter estimation workflow into three hierarchically coupled layers:

METHOD The three-layer optimization hierarchy

- Layer I (Observation Space): Raw 3D point cloud coordinates in millimetres.
- Layer II (Feature Primitive Estimation): Independent circle and arc fitting.
- Layer III (Global Line Model): Joint constrained optimization over the cone.

2.1 Layer I: The Observation Space $\mathcal{Y}$

The primary observations are extracted point coordinates from the 3D surface mesh of Dialface ObjID 18 (British Museum inv. 2547) [4].

Let $\mathcal{Y} = \{\mathbf{y}_{j,k}\}_{j=1, k=1}^{J, K_j}$ denote the set of J feature selections, where feature j contains K_j surface vertices. The measurement noise is modelled as zero-mean isotropic Gaussian noise with standard deviation $\sigma_0 \approx 0.3$ mm, corresponding to the photogrammetric noise floor.

2.2 Layer II: Feature-Level Primitive Regression

In Layer II, each feature band j is processed independently:

- **Hour Lines:** Fitted with 3D circular arcs:

$$S_{\text{arc}}(\mathbf{c}_j, r_j, \hat{\mathbf{n}}_j) = \sum_{k=1}^{K_j} (\|\mathbf{y}_{j,k} - \mathbf{c}_j\| - r_j)^2 + \lambda (\mathbf{y}_{j,k} \cdot \hat{\mathbf{n}}_j - d_n)^2. \quad (1)$$

Reading. The loss function S_{arc} penalizes deviations from the fitted circle radius r_j as well as out-of-plane distances along the normal $\hat{\mathbf{n}}_j$.

- **Day-Lines (Long Arcs):** Fitted with free planar circles to determine individual day-line planes and radii.

2.3 Layer III: Global Line Model and Joint Estimation

In Layer III, individual feature parameters are coupled into a unified global cone model.

Let $\boldsymbol{\theta} \in \mathbb{R}^7$ be the primary parameter vector:

$$\boldsymbol{\theta} = (A_x, A_y, A_z, \theta_C, \phi_C, \alpha, d)^\top, \quad (2)$$

where $\mathbf{A} = (A_x, A_y, A_z)$ is the 3D apex coordinate, (θ_C, ϕ_C) are the spherical angles parameterizing the unit axis $\hat{\mathbf{C}} \in S^2$, α is the cone half-angle, and d is the nodus distance.

The global objective function combines the orthogonal distance of all feature points to the cone surface:

$$S_{\text{global}}(\boldsymbol{\theta}) = \sum_{j=1}^J \sum_{k=1}^{K_j} d_{\perp}^2(\mathbf{y}_{j,k}; \boldsymbol{\theta}), \quad (3)$$

Reading. The global objective $S_{\text{global}}(\boldsymbol{\theta})$ sums the squared orthogonal Euclidean distances $d_{\perp}(\mathbf{y}_{j,k}; \boldsymbol{\theta})$ from every measured point $\mathbf{y}_{j,k}$ to the parametric cone surface.

For a point $\mathbf{y}$, the orthogonal distance to a cone with apex $\mathbf{A}$, axis $\hat{\mathbf{C}}$, and half-angle α is given in closed form by:

$$d_{\perp}(\mathbf{y}; \boldsymbol{\theta}) = \|\mathbf{y} - \mathbf{A}\| |\sin(\psi - \alpha)|, \quad \cos \psi = \frac{(\mathbf{y} - \mathbf{A}) \cdot \hat{\mathbf{O}}}{\|\mathbf{y} - \mathbf{A}\|}. \quad (4)$$

Reading. The orthogonal distance $d_{\perp}$ is the Euclidean distance from the point to the nearest generator on the cone nappe, where ψ is the angle between $(\mathbf{y} - \mathbf{A})$ and the cone opening vector $\hat{\mathbf{O}} = -\hat{\mathbf{C}}$.

3 Comparison of Optimization Algorithms

We evaluate four numerical optimization strategies across the BSDP corpus:

1. **Nelder–Mead Downhill Simplex:** A direct gradient-free heuristic. Robust to surface roughness but slow to converge near the global minimum.
2. **Trust Region Reflective (TRF):** A gradient-based trust-region method with bound constraints ($\alpha \in [10^\circ, 80^\circ]$, $d > 0$). Highly efficient for nonlinear least-squares.
3. **L-BFGS-B Joint Optimization:** A quasi-Newton method with limited-memory Hessian updates, scaling effectively to 33-parameter joint models.
4. **Direct Unlabelled Surface Fit (Orthogonal Least Squares):** Regressing the cone surface directly on the union of all surface vertices without requiring feature labels.

Table 1 presents the benchmark results for Dialface ObjID 18.

Table 1 Performance comparison of four optimization algorithms on Dialface ObjID 18 (BM inv. 2547).

Opti- mizer	Parameters	Iterations	Time [ms]	Residual RMS [mm]	Half-Angle α	Scale [mm]	d
Nelder– Mead	4	240	185	0.48	$34.12^\circ \pm 0.42^\circ$	100.4 ± 1.8	
TRF (Bounded)	7	38	42	0.39	$34.38^\circ \pm 0.11^\circ$	101.1 ± 0.6	
L-BFGS- B (Joint)	33	85	110	0.28	$34.40^\circ \pm 0.09^\circ$	101.3 ± 0.4	
Direct Surface OLS	6	24	28	0.39	$34.30^\circ \pm 0.08^\circ$	—	

4 Solving the Partial-Arc Degeneracy

A major failure mode in sundial reconstruction is the **partial-arc bias**. When a seasonal day-line is only partially preserved (such as the winter solstice line on ObjID 18, which spans only $\Delta\theta \approx 102^\circ$ with 9 centreline points), unconstrained planar circle fitting yields a plane normal that is tilted by over 18° relative to the true polar axis.

CONCEPT Resolution via coaxial constrained estimation

On short fragments, free-plane circle fitting is mathematically degenerate. The robust solution is two-stage estimation:

1. Fit the global cone axis $\hat{C}$ directly to all unlabelled cavity points.
2. Fit the fragmented day-line as a circle constrained to be strictly coaxial with $\hat{C}$.

On ObjID 18, coaxial constrained fitting reduces the winter day-line radius uncertainty from ± 8.4 mm to ± 1.78 mm, restoring exact geometric compatibility with the equinoctial and summer circles.

5 Parameter Uncertainties and the Jacobian Covariance Matrix

Following Bard (1974), parameter uncertainties are derived from the Jacobian matrix $J \in \mathbb{R}^{N \times P}$ evaluated at the optimal estimate $\hat{\theta}$:

$$J_{ij} = \left. \frac{\partial r_i(\theta)}{\partial \theta_j} \right|_{\theta=\hat{\theta}}. \quad (5)$$

Reading. The Jacobian entry J_{ij} is the partial derivative of the i -th geometric residual with respect to the j -th structural parameter.

The asymptotic parameter covariance matrix V_θ is:

$$V_\theta = s^2 (J^\top J)^{-1}, \quad s^2 = \frac{S(\hat{\theta})}{N - P}, \quad (6)$$

Reading. The covariance matrix V_θ equals the estimated residual variance s^2 multiplied by the inverse of the normal equation matrix $J^\top J$, where N is the number of observations and P is the number of parameters.

The 1σ standard errors are the square roots of the diagonal entries:

$$\sigma(\theta_j) = \sqrt{V_{jj}}. \quad (7)$$

Reading. The standard uncertainty of each parameter is the square root of its corresponding diagonal element in the covariance matrix.

For Dialface ObjID 18, this yields the rigorous parameter estimates: $\alpha = 34.40^\circ \pm 0.09^\circ$, $d = 101.3 \pm 0.4$ mm, and axis inclination $\varphi = 41.85^\circ \pm 0.28^\circ$.

6 Autonomous Agent Execution Box

When integrated into an AI-orchestrated scientific discovery platform, the nonlinear optimization engine is invoked by sub-agents to calibrate feature detectors across heterogeneous stone scans.

OPERATION Nonlinear Cone Model Optimization		0.12 s
Q Estimate 7-parameter cone and compute Jacobian covariance for ObjID 18		
1	data-prep	Loads 3,345 extracted feature points and initializes parameter vector via algebraic Kåsa fit
2	direct-surface-fit	Executes orthogonal distance regression to find initial apex A_0 and axis $\hat{C}_0$
3	trf-optimizer	Runs Trust Region Reflective optimization with bounds $\alpha \in [20^\circ, 60^\circ]$, $d > 0$
4	jacobian-evaluator	Computes $J^T J$ and inverts parameter covariance matrix V_θ
5	coaxial-projector	Projects winter day-line onto coaxial constraint, eliminating partial-arc bias
→ $\alpha = 34.40^\circ \pm 0.09^\circ$, $d = 101.3 \pm 0.4$ mm, residual RMS 0.39 mm, full covariance matrix verified		

7 Conclusion

Recovering the geometry of ancient conic sundials requires nonlinear parameter estimation methods that respect the physical morphology of the stone and the rotational constraints of gnomonics. By structuring the problem into an explicit three-layer estimation hierarchy, resolving partial-arc bias through coaxial constraints, and computing parameter covariances via the Jacobian, feature detectors can be calibrated with sub-millimetre precision across diverse archaeological collections.

References

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