

Construction Geometry of the Conic Sundial with Horizontal Gnomon

An Analytical Framework for Shadow Projection and Apex-Free Parameter Recovery on an Inclined Cone

Gerd Graßhoff 

Humboldt-Universität zu Berlin, Deutsches Museum
gerd.grasshoff@hu-berlin.de

ABSTRACT

A conic sundial with its nodus on the polar axis is determined by three construction choices: the intended latitude (the *klima* proportion of the ancient shadow-table tradition), the cone opening, and an arbitrary scale. Because the nodus is mounted directly on the celestial rotation axis, every fixed solar declination projects a circular shadow path onto the interior cone surface, perpendicular to that axis. This paper establishes the complete analytical geometry of that instrument in a unified East–North–Up world frame and derives its exact closed-form inversion. The central theoretical result is that the three seasonal meridian marks (P_S, P_E, P_W) determine the cone — half-angle α , scale d , and apex position A — independently of geographical latitude φ . Latitude is not encoded in the relative positions of the noon marks; it enters the artifact exclusively through the hour division of the solstitial day-lines and the dressing of the block faces. A closed-form marking ratio $Q = (L_S - L_W)/(L_S - L_E)$ recovers α and the apex even when the vertex is chipped or missing. The mathematical framework is verified against an exact synthetic Rhodes construction and compared with measured artifacts from the Berlin Sundial Project corpus.

KEYWORDS conic sundial · gnomonics · shadow geometry · klimata · ancient instruments · analytical inversion · apex-free reconstruction

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i Supplement S2

Supplement S2 to the Ancient Sundials repository (Berlin Sundial Collaboration, ancient-sundials.lettres.ai). This draft documents the shadow animation and the astronomical picture of each object (noon altitude, day length). It is a preprint draft and has not been peer reviewed. Where a figure in the repository differs from a figure here, the repository, which is rebuilt from the current database, is authoritative.

1 Introduction: The Geometry of Ancient Timekeeping

Ancient Greek and Roman horology produced three principal types of spherical and conical shadow-receiving dials [1], [2]. Among these, the conical sundial with a horizontal gnomon occupies a privileged mathematical position. Unlike planar sundials, where seasonal shadow paths are hyperbolae or ellipses varying in curvature across the year, the conic dial with a nodus on the polar axis projects every seasonal day-curve as an exact circular arc.

When archaeologists excavate a fragmentary sundial in the Mediterranean basin, the stone is frequently damaged: the apex of the hollowed cone is chipped, the gnomon has perished, and the original orientation on its pedestal is lost. The computational task is to recover the intended construction parameters from the 3D scan of the surviving stone fragment.

CONCEPT The central thesis in one sentence

The three seasonal noon marks on a conic sundial determine the cone half-angle and the nodus distance completely and independently of latitude; the latitude φ is not recorded in the noon marks and must be recovered from the hour-line network or from the dressed reference planes of the block.

This thesis directly corrects an assumption in the literature that treats the solstitial noon-mark ratio as a latitude diagnostic. On a cone whose axis is aligned with the celestial pole, rotational symmetry uncouples the axial spacing of the seasonal day-lines from the tilt of the cone axis above the horizon. The noon marks record the shape of the cone; the hour network records the latitude.

Five contributions establish this framework:

1. **Forward Coordinate Geometry:** A rigorous formulation of shadow projection in an East–North–Up (ENU) world frame and an equatorial polar frame, yielding closed-form ray–cone intersections.
2. **Day-Line Circularity:** A formal proof that all seasonal shadow traces are planar circles orthogonal to the cone axis, with the equinoctial circle sitting at an axial distance identical to the nodus distance (d).
3. **Constructor’s Proportion Vocabulary:** A complete formulation of the instrument’s attributes in the proportional language of ancient dialmakers, governed by the workshop rule $x = \tan \alpha \tan \varepsilon$.
4. **Apex-Free Closed-Form Inversion:** The derivation of the marking ratio Q , which computes the cone half-angle α , the apex location A , and the scale d directly from three collinear meridian marks without requiring the physical apex.
5. **Marking Sufficiency & Sensitivity:** A comprehensive classification of which subsets of surface features determine which structural parameters, together with analytical error bounds.

Direct physical evidence for this proportional design method comes from the unfinished conic sundial discovered on the Agora of the Italians on Delos [3]. The Delos artifact demonstrates that ancient artisans committed the latitude proportion to the stone block before hollowing the cone, scribed reference transfer circles, and incised the noon line by simple geometric ratios.

2 Physical Construction Elements

The geometry of the instrument is established by one fixed shadow-casting point and one receiving surface of revolution.

CONCEPT The Nodus N

The shadow-casting point of the sundial: a small metallic sphere mounted on the Earth's polar axis, situated exactly on the observer's horizontal plane. All solar shadow rays originate geometrically from this point.

CONCEPT The Cone, Apex A , and Half-Angle α

The receiving dialface is the interior nappe of a right circular cone. Its axis of rotational symmetry is collinear with the Earth's polar axis. The apex A is positioned on this axis at a distance d above the nodus, and the cone opens southward and downward with half-angle α .

CONCEPT The Gnomon

A rigid horizontal support rod lying in the observer's meridian plane, extending northward from the nodus along the horizon to anchor into the stone block. Only the nodus, not the support rod, casts the time-telling shadow point.

Because the nodus N lies on the horizontal plane ($Z = 0$), the nodus shadow first strikes the receiving cone at the exact astronomical moment of sunrise, sweeps across the inscribed network during daylight, and leaves the cone at sunset.

Figure 1 illustrates the meridian cross-section of the instrument for a construction at Rhodes ($\varphi = 36^\circ$, $\alpha = 45^\circ$).

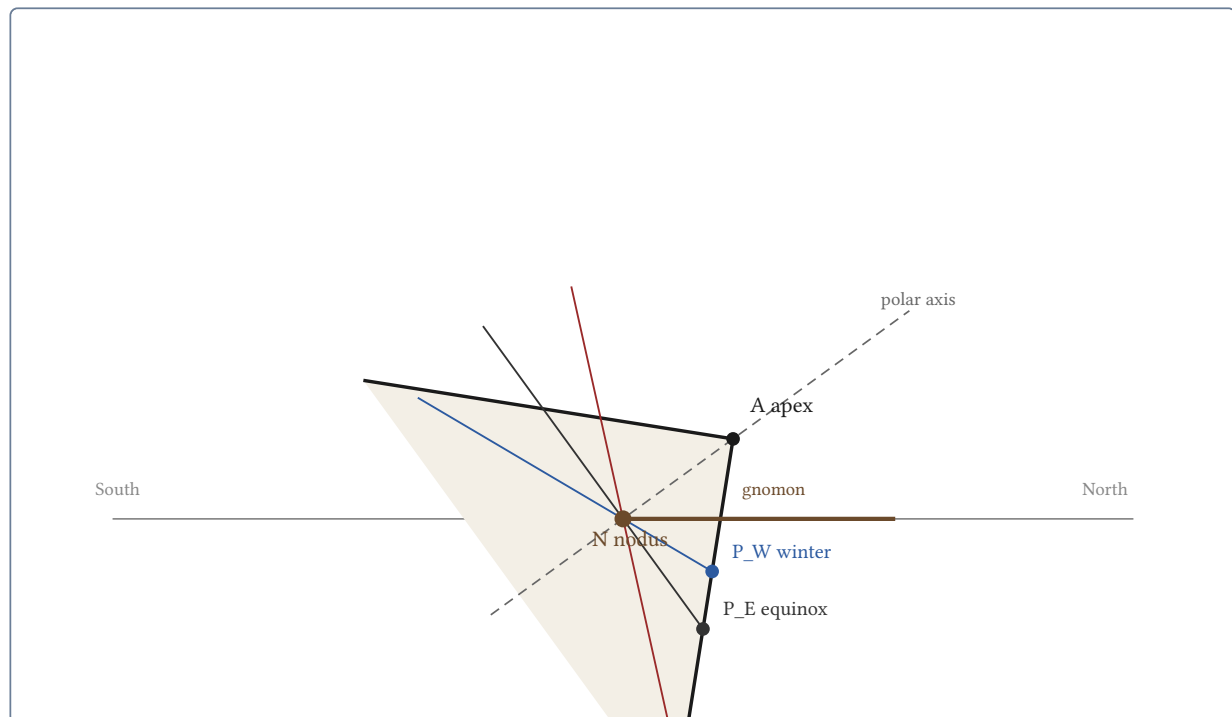


Figure 1 Meridian cross-section of a conic sundial constructed for Rhodes ($\varphi = 36^\circ$, $\alpha = 45^\circ$). View from the East (South is to the left, North to the right). The nodus N rests on the horizon ($Z = 0$). The polar axis is inclined at angle φ . The cone apex A lies on the polar axis at distance d above N . The cone opens southward at half-angle α . Three noon rays passing through N strike the lower meridian generator at P_W (winter solstice), P_E (equinox), and P_S (summer solstice).

Three independent quantities fully determine the construction:

1. **Latitude** φ — the celestial tilt of the polar axis above the horizon, fixed by the geographical location.
2. **Nodus Distance** d — the distance along the polar axis from the apex A to the nodus N . This is the constructor's arbitrary scale factor.
3. **Cone Opening** $\rho = \tan \alpha$ — the ratio of rim radius to cone height, chosen by the artisan for aesthetic and structural reasons.

The only physical constraint on the scale pair (d, H) , where H is the cone height along the axis, is that the nodus must sit inside the cone cavity:

$$d < H. \quad (1)$$

Reading. The nodus distance d along the axis must be strictly less than the total axial depth H of the cone cavity, ensuring that shadows fall upon the interior stone surface throughout the year.

3 Coordinate Frames and Transformations

To model the instrument unambiguously, we define two right-handed Cartesian coordinate systems.

3.1 The World Frame (ENU)

The world coordinate frame is fixed to the local observer with the origin at the physical nodus N :

- $+X_w$ points **East** (orthogonal to the meridian plane),
- $+Y_w$ points **North** (along the horizontal meridian),
- $+Z_w$ points **Up** (perpendicular to the local horizon).

In this frame, the unit vector along the polar axis pointing toward the North Celestial Pole is:

$$\hat{\mathbf{C}} = (0, \cos \varphi, \sin \varphi). \quad (2)$$

Reading. The cone axis $\hat{\mathbf{C}}$ has zero East–West component, a horizontal North component equal to $\cos \varphi$, and a vertical Up component equal to $\sin \varphi$, corresponding to an elevation angle of φ above the northern horizon.

The cone opens in the opposite direction, toward the South and downward:

$$\hat{\mathbf{O}} = -\hat{\mathbf{C}} = (0, -\cos \varphi, -\sin \varphi). \quad (3)$$

Reading. The cone opening vector $\hat{\mathbf{O}}$ points southward with component $-\cos \varphi$ and downward with component $-\sin \varphi$.

3.2 The Equatorial Polar Frame

The polar coordinate frame has its $+Z_p$ axis aligned with $\hat{\mathbf{C}}$. The rotation matrix R transforming coordinates from the polar frame to the world frame ($\mathbf{v}_w = R \mathbf{v}_p$) is uniquely determined by mapping the polar axis $(0, 0, 1)$ to $\hat{\mathbf{C}}$ and mapping the equinoctial sunrise direction $(-1, 0, 0)$ to the East vector $(1, 0, 0)$:

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sin \varphi & \cos \varphi \\ 0 & \cos \varphi & \sin \varphi \end{pmatrix}. \quad (4)$$

Reading. The rotation matrix R is an orthonormal transformation with determinant $+1$, mapping polar equatorial vectors into the local East–North–Up world coordinates.

3.3 The Mathematical Frame (Apex at Origin)

For analytical derivations of ray–cone intersections, it is mathematically convenient to translate the origin to the cone apex A . In this mathematical frame:

$$\mathbf{A} = (0, 0, 0), \quad \mathbf{N} = d \hat{\mathbf{O}} = (0, -d \cos \varphi, -d \sin \varphi). \quad (5)$$

Reading. In the mathematical frame, the cone apex A is positioned at the coordinate origin, and the nodus N is located at distance d along the opening direction $\hat{O}$.

4 Shadow Projection Algebra

Let the Sun's position in the equatorial polar frame be defined by its solar declination δ and hour angle h (measured positive westward from the meridian, with $h = 0$ at solar noon):

$$\hat{S}_p = (\cos \delta \sin h, \cos \delta \cos h, \sin \delta). \quad (6)$$

Reading. The solar direction unit vector $\hat{S}_p$ is decomposed into equatorial coordinates with declination δ and hour angle h .

In the world frame, the Sun's direction is $\hat{S}_w = R \hat{S}_p$:

$$\hat{S}_w = \begin{pmatrix} -\cos \delta \sin h \\ -\sin \varphi \cos \delta \cos h + \cos \varphi \sin \delta \\ \cos \varphi \cos \delta \cos h + \sin \varphi \sin \delta \end{pmatrix}. \quad (7)$$

Reading. The world-frame solar vector $\hat{S}_w$ specifies the direction of incoming sunlight; its vertical Z -component is the sine of the solar altitude angle $a_\odot$.

The shadow ray originates at the nodus N and propagates in the direction $-\hat{S}_w$:

$$\mathbf{P}(k) = \mathbf{N} - k \hat{S}_w, \quad k > 0. \quad (8)$$

Reading. A point $\mathbf{P}(k)$ along the shadow ray is obtained by starting at the nodus N and moving a positive scalar distance k away from the Sun along the direction $-\hat{S}_w$.

A point $\mathbf{P}$ lies on the surface of a cone with vertex at the origin, axis $\hat{C}$, and half-angle α if and only if:

$$(\mathbf{P} \cdot \hat{C})^2 = |\mathbf{P}|^2 \cos^2 \alpha, \quad \mathbf{P} \cdot \hat{C} < 0. \quad (9)$$

Reading. The squared scalar projection of the vector $\mathbf{P}$ onto the axis $\hat{C}$ equals the squared length of $\mathbf{P}$ multiplied by $\cos^2 \alpha$, subject to the condition that $\mathbf{P} \cdot \hat{C} < 0$ to select the south-opening nappe.

Substituting Equation 8 into Equation 9 yields a quadratic equation in the ray parameter k :

$$A_k k^2 + 2 B_k k + C_k = 0, \quad (10)$$

where, noting that $\mathbf{N} = d \hat{O} = -d \hat{C}$ (so $\mathbf{N} \cdot \hat{C} = -d$ and $|\mathbf{N}|^2 = d^2$), the coefficients are:

$$\begin{aligned} A_k &= (\hat{S}_w \cdot \hat{C})^2 - \cos^2 \alpha = \sin^2 \delta - \cos^2 \alpha, \\ B_k &= d(\hat{S}_w \cdot \hat{C} - \cos^2 \alpha (\hat{O} \cdot \hat{S}_w)) = d \sin \delta (1 + \cos^2 \alpha) \dots \\ C_k &= d^2(1 - \cos^2 \alpha) = d^2 \sin^2 \alpha. \end{aligned} \quad (11)$$

Reading. The intersection of the shadow ray with the cone surface reduces to a standard quadratic equation whose positive root k yields the unique physical shadow point on the interior nappe.

5 The Day-Line Circularity Theorem

A seasonal *day-line* is the continuous curve traced by the nodus shadow on the cone as the Sun moves from sunrise (h_{rise}) to sunset (h_{set}) at fixed declination δ .

METHOD Theorem: Day-Line Circularity

For any fixed solar declination δ , the shadow locus on the interior surface of a conic sundial with its nodus on the polar axis is an exact planar circle orthogonal to the cone axis $\hat{C}$.

Proof. The nodus N lies on the axis of rotational symmetry $\hat{C}$. For a fixed solar declination δ , the diurnal path of the Sun is a great or small circle centered on the celestial polar axis. The entire physical system — the light source path, the shadow-casting nodus, and the cone surface — is strictly invariant under rotations about $\hat{C}$.

Let $P(h)$ be the shadow point at hour angle h . A change in hour angle Δh corresponds to a pure rigid rotation of the incident ray about $\hat{C}$ by angle Δh . Because the cone is a surface of revolution about $\hat{C}$, the resulting intersection point $P(h + \Delta h)$ is obtained simply by rotating $P(h)$ by Δh around $\hat{C}$. Therefore, the axial distance $\ell(\delta) = -\mathbf{P} \cdot \hat{C}$ and the radial distance $r(\delta) = |\mathbf{P} \times \hat{C}|$ are strictly constant for all $h \in [h_{\text{rise}}, h_{\text{set}}]$. A locus of constant axial position and constant radius on a cone is an exact circle lying in a plane perpendicular to $\hat{C}$. ■

Figure 2 displays the geometry of the cone surface unfolded into a flat plane.

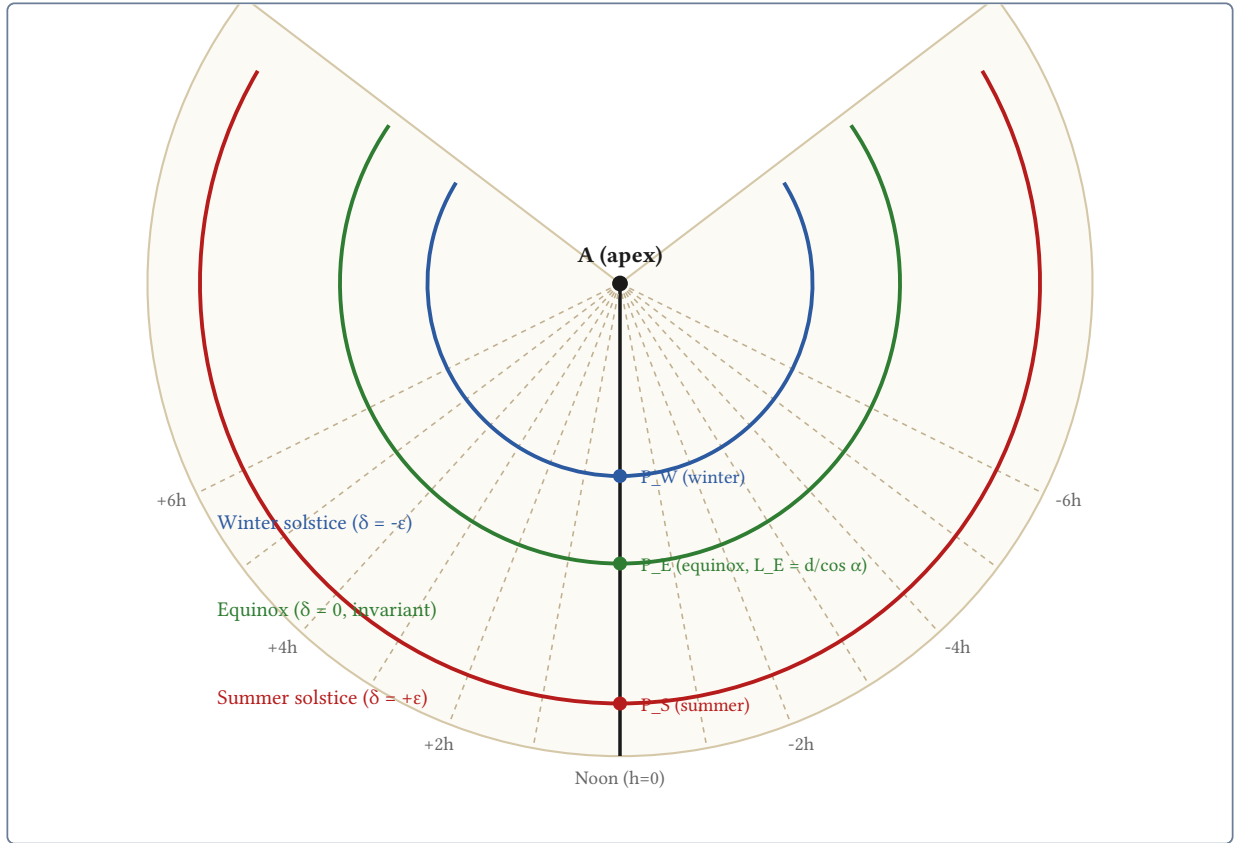


Figure 2 Unfolded cone surface for half-angle $\alpha = 45^\circ$ (sector angle $2\pi \sin 45^\circ \approx 254.6^\circ$). The apex A is at the top center. The central vertical line is the noon meridian generator ($h = 0$). Straight radial lines represent hour lines. The three seasonal day-lines — winter solstice (blue, inner), equinox (green, middle), and summer solstice (red, outer) — unfold to exact concentric circular arcs centered on A .

6 Proportions and the Constructor's Vocabulary

Ancient sundial makers did not specify their instruments using decimal degree angles. They operated in a coherent language of geometric proportions, expressing shadow lengths, cone openings, and day-line positions as ratios.

6.1 The Equinox Invariant

Evaluating the shadow projection at the astronomical equinox ($\delta = 0$):

$$\ell_E \equiv -\mathbf{P}(\delta = 0) \cdot \hat{C} = d. \quad (12)$$

Reading. At the equinox ($\delta = 0$), the sunlight vector is strictly perpendicular to the polar axis ($\hat{\mathbf{S}}_w \cdot \hat{\mathbf{C}} = \sin 0 = 0$). The shadow ray cast by the nodus lies in the plane perpendicular to the axis passing through the nodus itself. Consequently, the equinoctial day-line circle sits at an axial distance ℓ_E from the apex that is identically equal to the nodus distance d , regardless of the cone half-angle α or latitude φ .

The corresponding slant distance L_E measured along the meridian generator on the stone surface is:

$$L_E = \frac{\ell_E}{\cos \alpha} = \frac{d}{\cos \alpha}. \quad (13)$$

Reading. The slant distance L_E from the apex to the equinoctial noon mark equals the nodus distance d divided by the cosine of the cone half-angle α .

6.2 The Workshop Rule x and Solstitial Spacings

For any solar declination δ , the axial distance $\ell(\delta)$ from the apex to the noon shadow mark on the meridian is given by:

$$\ell(\delta) = \frac{d}{1 - \tan \alpha \tan \delta}. \quad (14)$$

Reading. The axial distance $\ell(\delta)$ equals the nodus distance d divided by the quantity $1 - \tan \alpha \tan \delta$.

Evaluating Equation 14 at the summer solstice ($\delta = +\varepsilon$) and winter solstice ($\delta = -\varepsilon$), we define the fundamental dimensionless **workshop ratio**:

$$x := \tan \alpha \tan \varepsilon. \quad (15)$$

Reading. The workshop ratio x is the product of the tangent of the cone half-angle α and the tangent of the obliquity of the ecliptic ε .

The solstitial axial distances become:

$$\ell_S = \frac{d}{1 - x}, \quad \ell_W = \frac{d}{1 + x}. \quad (16)$$

Reading. The summer solstice noon mark sits at axial distance $d/(1 - x)$ from the apex (deepest into the cone), while the winter solstice noon mark sits at axial distance $d/(1 + x)$ (closest to the apex).

The three seasonal noon marks on the meridian generator thus satisfy the celebrated **noon-mark triple**:

$$\ell_W : \ell_E : \ell_S = \frac{1}{1 + x} : 1 : \frac{1}{1 - x}. \quad (17)$$

Reading. The ratio of winter, equinox, and summer axial noon marks depends exclusively on the single parameter x . For the standard Hellenistic workshop choice $x = 1/4$ (obtained, for example, with $\alpha = 30^\circ$ and $\varepsilon \approx 23.5^\circ$), this ratio yields the integers $12 : 15 : 20$.

Table 1 summarizes the complete attribute vocabulary of the conic dial.

Table 1 The attribute vocabulary of the ancient conic sundial. Every entry is a proportion or an integer count; degree values are modern conversions.

Attribute	Historical Notation	Geometric Meaning
Gnomon standard	$g = 60$	Sexagesimal parts of the vertical gnomon
Klima ratio	$\sigma_E = s_{\text{eq}} : g$	Equinoctial noon shadow; $\sigma_E = \tan \varphi$
Solstitial shadow ratios	σ_S, σ_W	$\sigma_S = \tan(\varphi - \varepsilon)$, $\sigma_W = \tan(\varphi + \varepsilon)$
Cone opening ratio	$\rho = R_{\text{rim}} : H$	$\rho = \tan \alpha$; ratio of rim radius to cone height
Nodus distance	d	Apex-to-nodus distance along axis; scale unit
Equinox invariant	$\ell_E : d = 1 : 1$	Equinoctial circle sits at distance d from apex
Workshop rule	$x = \tan \alpha \tan \varepsilon$	Dimensionless factor governing solstitial marks
Noon-mark triple	$\ell_W : \ell_E : \ell_S$	Triplet of axial noon distances: $(1+x)^{-1} : 1 : (1-x)^{-1}$
Equinoctial division	$H_E = 90^\circ$	Quarter-circle in 6 seasonal hours ($15^\circ/\text{h}$); calibration invariant
Solstitial division	$H(\pm\varepsilon)$	$\cos H = -\tan \varphi \tan(\pm\varepsilon)$; where φ enters the network

7 The Klimata Tradition and the Vertical Gnomon

Greek mathematical geography divided the inhabited world into latitudinal zones called *klimata*, defined by the length of the longest day and the shadow cast by a vertical gnomon of $g = 60$ parts at noon [4], [5].

Figure 3 illustrates the vertical gnomon shadow triangle for Alexandria.

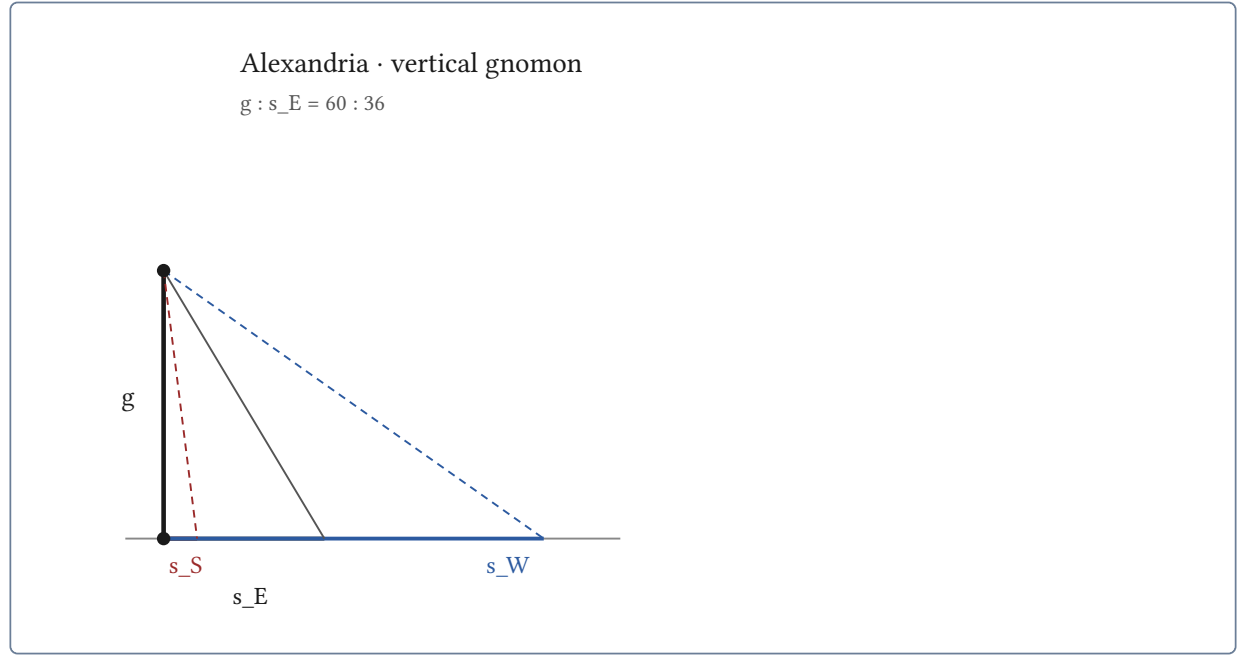


Figure 3 The vertical gnomon shadow triangle for Alexandria ($\varphi = 30^\circ 58'$). A vertical gnomon of height $g = 60$ parts casts noon shadows of length $s_S = 7; 29$ (summer solstice), $s_E = 36; 00$ (equinox), and $s_W = 85; 08$ (winter solstice). The equinoctial ratio is exactly $g : s_E = 60 : 36 = 5 : 3$.

The vertical gnomon and the conic sundial represent dual geometric realisations of the same astronomical quantities:

- In the **vertical gnomon**, the receiving surface is horizontal ($Z = 0$), and the latitude φ is recorded directly in the equinoctial shadow length $s_E = g \tan \varphi$.
- In the **conic sundial**, the receiving surface is an inclined cone whose axis is tilted at angle φ . The equinoctial mark on the cone is invariant ($\ell_E = d$), and the latitude is recorded in the spatial orientation of the cone axis and the angular span of the solstitial hour network.

Table 2 lists the canonical shadow lengths across the seven ancient klimata computed using Ptolemy's obliquity $\varepsilon = 23^\circ 51' 20''$ [5, Book II].

Table 2 Canonical noon shadow lengths for a vertical gnomon of $g = 60$ parts, computed from Ptolemy's klimata table ($\varepsilon = 23^\circ 51' 20''$). Sexagesimal parts ($p; q = p + q/60$).

Location	Klima	Latitude φ	Equinox s_{eq}	Summer s_{su}	Winter s_{wi}	$\sigma_E = s_{eq}/g$
Syene	II	$23^\circ 51'$	26; 32	0; 00 (zenith)	65; 57	0.442
Alexan- dria	III	$30^\circ 58'$	36; 00	7; 29	85; 08	0.600
Rhodes	IV	$36^\circ 00'$	43; 36	12; 55	103; 19	0.727
Delos	—	$37^\circ 24'$	45; 52	14; 27	109; 23	0.765
Athens	—	$37^\circ 58'$	46; 49	15; 05	112; 00	0.780
Helle- spont	V	$40^\circ 56'$	52; 02	18; 26	127; 27	0.867
Rome	—	$41^\circ 54'$	53; 50	19; 33	133; 14	0.897
Massalia	VI	$43^\circ 05'$	56; 07	20; 56	140; 56	0.935

8 Inversion: From Surface Marks to Construction Parameters

When reconstructing an ancient sundial from a 3D scan, we face the inverse problem: given the coordinates of surviving engraved marks on the stone, recover (α, d, φ, A) .

8.1 Classical Inversion (Apex A Preserved)

If the apex A is physically preserved, parameter recovery proceeds directly:

1. Measure slant distances from A to the noon marks: $L_S = \| \mathbf{P}_S - A \|$, $L_E = \| \mathbf{P}_E - A \|$, $L_W = \| \mathbf{P}_W - A \|$.
2. Compute the solstitial slant ratio:

$$\frac{L_S}{L_W} = \frac{\ell_S}{\ell_W} = \frac{1+x}{1-x} = \frac{1+\tan \alpha \tan \varepsilon}{1-\tan \alpha \tan \varepsilon}. \quad (18)$$

Reading. The ratio of summer to winter slant distances depends solely on α and ε , completely independent of latitude φ .

3. Invert Equation 18 for the cone half-angle:

$$\tan \alpha = \frac{1}{\tan \varepsilon} \left(\frac{L_S/L_W - 1}{L_S/L_W + 1} \right). \quad (19)$$

Reading. The tangent of the half-angle equals the normalized difference-to-sum ratio of the solstitial slant marks divided by $\tan \varepsilon$.

4. Compute the nodus distance using the equinox invariant:

$$d = L_E \cos \alpha. \quad (20)$$

Reading. The nodus distance d equals the equinoctial slant distance L_E multiplied by $\cos \alpha$.

Table 3 lists the slant ratio L_S/L_W as a function of half-angle α at $\varepsilon = 23.44^\circ$.

Table 3 Solstice slant-distance ratio L_S/L_W as a function of cone half-angle α ($\varepsilon = 23.44^\circ$).

Half-An- gle α	20°	25°	30°	35°	40°	45°	50°	55°	60°
Ratio L_S/L_W	1.37	1.50	1.67	1.88	2.14	2.53	3.14	4.19	7.03

8.2 Apex-Free Inversion via the Marking Ratio Q

On damaged fragments where the apex A is broken away, the three seasonal noon marks P_S, P_E, P_W alone suffice to recover α, A , and d in closed form.

We define the dimensionless **marking ratio** Q :

$$Q := \frac{L_S - L_W}{L_S - L_E}. \quad (21)$$

Reading. Q is the ratio of the total solstitial distance ($L_S - L_W$) to the summer-to-equinox distance ($L_S - L_E$) along the meridian generator.

Expressing the slant differences in terms of d, α , and $x = \tan \alpha \tan \varepsilon$:

$$L_S - L_W = \frac{d}{\cos \alpha} \left(\frac{1}{1-x} - \frac{1}{1+x} \right) = \frac{2dx}{\cos \alpha (1-x^2)}, \quad (22)$$

$$L_S - L_E = \frac{d}{\cos \alpha} \left(\frac{1}{1-x} - 1 \right) = \frac{dx}{\cos \alpha (1-x)}. \quad (23)$$

Dividing Equation 22 by Equation 23:

$$Q = \frac{2}{1+x} = \frac{2}{1+\tan \alpha \tan \varepsilon}. \quad (24)$$

Reading. The marking ratio Q is a simple rational function of the workshop parameter x , completely independent of the scale d and latitude φ .

Solving Equation 24 explicitly for α :

$$\alpha = \arctan\left(\frac{2-Q}{Q \tan \varepsilon}\right), \quad 1 < Q < 2. \quad (25)$$

Reading. The cone half-angle α is the inverse tangent of $(2-Q)$ divided by the product $Q \tan \varepsilon$.

CONCEPT Physical Scope of the Q -Formula

The mathematical condition $1 < Q < 2$ corresponds to $0 < x < 1$, which is the exact physical condition that the cone opens ($\alpha > 0^\circ$) and that the summer solstice rays strike the interior nappe ($\alpha < 90^\circ - \varepsilon$).

8.2.1 The Step-by-Step Apex-Free Algorithm

1. **Generator Direction:** Compute the unit direction vector along the meridian from P_S toward P_W (inward toward the apex):

$$\hat{\mathbf{g}} = \frac{\mathbf{P}_W - \mathbf{P}_S}{\|\mathbf{P}_W - \mathbf{P}_S\|}. \quad (26)$$

Reading. $\hat{\mathbf{g}}$ is the unit vector pointing along the meridian line from the summer mark toward the winter mark.

2. **Projected Segment Lengths:** Compute the total baseline g_{len} and the projected equinox distance t_E :

$$g_{\text{len}} = \|\mathbf{P}_W - \mathbf{P}_S\|, \quad t_E = (\mathbf{P}_E - \mathbf{P}_S) \cdot \hat{\mathbf{g}}. \quad (27)$$

Reading. g_{len} is the physical distance between summer and winter marks; t_E is the projected distance from summer to equinox. Collinearity is verified if the orthogonal residual $\|\mathbf{P}_E - (\mathbf{P}_S + t_E \hat{\mathbf{g}})\|$ is within scan noise ($\lesssim 0.3$ mm).

3. **Recover Half-Angle α :** Compute $Q = g_{\text{len}}/t_E$ and evaluate $\alpha = \arctan((2-Q)/(Q \tan \varepsilon))$.
4. **Locate the Apex A :** The slant ratio $r_S = L_S/L_E = 1/(1-x)$ gives the apex distance parameter t_A :

$$t_A = \frac{r_S t_E}{r_S - 1} = L_S, \quad A = \mathbf{P}_S + t_A \hat{\mathbf{g}}. \quad (28)$$

Reading. The distance t_A along $\hat{\mathbf{g}}$ from P_S places the true apex A in 3D space.

5. **Recover Scale d :**

$$L_E = t_A - t_E, \quad d = L_E \cos \alpha. \quad (29)$$

Reading. Subtracting t_E from L_S gives L_E , whose product with $\cos \alpha$ yields the scale d .

8.3 Independent Latitude Recovery (ε -Free)

Latitude φ is recovered independently of the noon marks by pairing the radial placement x of a solstitial day-line with its measured half-day hour angle H :

$$\tan \varphi = \tan \alpha \frac{|\cos \tilde{H}|}{|x|}. \quad (30)$$

Reading. The tangent of latitude φ equals $\tan \alpha$ multiplied by the ratio of the absolute cosine of the unit-calibrated half-day arc $\tilde{H}$ to the placement parameter x . The realized epoch obliquity $\tan \delta$ cancels out completely, rendering the latitude recovery robust against epoch uncertainties.

8.4 Marking Sufficiency

Table 4 catalogs which parameters are determinable from various observable feature sets.

Table 4 Marking sufficiency catalog for conic sundial reconstruction.

Available Set	Marking	Recoverable Parameters	Undetermined Parameters	Paras	Analytical Method
$\{P_S, P_E\}$		Meridian generator line $\hat{g}$, segment $L_S - L_E$	α, d, φ, A		Linear interpolation
$\{P_S, P_E, P_W\}$		Q , half-angle α , apex A , slants L_S, L_E, L_W , scale d	Latitude φ , axis tilt		Q -inversion (Equation 25–Equation 29)
$\{P_S, P_E, P_W\} + \geq 3$ hour marks		Complete geometry: $\alpha, d, A, \hat{C}, \varphi$	None (complete reconstruction)		Q -inversion + 3D circle fit
$\{A, P_S, P_E, P_W\}$		α, d, L_S, L_E, L_W	Latitude φ		Classical inversion (Equation 19)
Dressed seat & upper front face		Cone axis tilt φ , face orientation	Scale d , half-angle α		Plane normal regression

9 Synthetic Validation: A Rhodes Construction

We validate the apex-free algorithm on an exact synthetic model constructed for Rhodes ($\varphi = 36.00^\circ$, $\alpha = 45.00^\circ$, $d = 100.00$ mm, $\varepsilon = 23.44^\circ$).

1. Forward Generation:

- $x = \tan 45^\circ \tan 23.44^\circ = 0.433568$.
- Axial marks: $\ell_S = 176.544$ mm, $\ell_E = 100.000$ mm, $\ell_W = 69.756$ mm.
- Slant marks ($L = \ell / \cos 45^\circ$): $L_S = 249.670$ mm, $L_E = 141.421$ mm, $L_W = 98.650$ mm.

2. Segment Input:

- $g_{\text{len}} = L_S - L_W = 151.020$ mm.
- $t_E = L_S - L_E = 108.249$ mm.

3. Inverse Recovery:

- Marking ratio: $Q = 151.020/108.249 = 1.395121$.
- Half-angle:

$$\alpha = \arctan\left(\frac{2 - 1.395121}{1.395121 \times \tan 23.44^\circ}\right) = \arctan(1.000000) = 45.00^\circ.$$

- Scale recovery: $r_S = 1/(1 - 0.433568) = 1.765436$; $t_A = (1.765436 \times 108.249)/0.765436 = 249.670$ mm = L_S .
- Equinox slant: $L_E = 249.670 - 108.249 = 141.421$ mm.
- Nodus distance: $d = 141.421 \times \cos 45^\circ = 100.000$ mm.

Table 5 confirms exact numerical closure.

Table 5 Numerical verification of closed-form parameter recovery on the synthetic Rhodes model.

Parameter	Nominal Truth	Ground	Inverted Estimate	Absolute Error
Half-angle α	45.0000°		45.0000°	$< 10^{-6}^\circ$
Marking ratio Q	1.395121		1.395121	$< 10^{-6}$
Slant ratio L_S/L_W	2.530873		2.530873	$< 10^{-6}$
Nodus scale d	100.000 mm		100.000 mm	$< 10^{-5}$ mm

10 What Archaeological Stones Add

Applying this analytical geometry to scanned archaeological stones introduces real-world physical effects documented in the companion studies of ObjID 18 (British Museum inv. 2547) [6], [7] and ObjID 126 (Delos) [8]:

1. **Engraved Hour Lines as Circular Arcs:** On ObjID 18, the engraved hour lines are circular arcs with radii between 113 mm and 280 mm (fit RMS 0.20–0.71 mm), rather than mathematical generators (which leave bow residuals of 2.2–7.6 mm). The straight line of the theoretical cone represents the geometric ideal; the circular arc is the artisan’s physical realization.
2. **Groove Centreline Extraction:** At photogrammetric scan resolutions, the V-shaped chisel notch of worn engravings lies within the ± 0.3 mm surface noise floor. Centreline extraction must be performed by medial-line filtering across selected feature bands rather than unconstrained depth minimization.
3. **Construction Error Attribution:** On ObjID 18, the winter hour division and cone axis yield a consensus latitude of $\varphi = 41.45^\circ \pm 0.30^\circ$ ($s = 53; 02$ for $g = 60$, corresponding to Ptolemy’s Hellespont/Rome klimata), while the summer hour division deviates at 14σ — an isolated ancient artisan error correctly identified by leave-one-out diagnostics rather than obscured by global averaging.

11 Constraint Catalog

Table 6 summarizes the complete set of formal geometric constraints governing the conic sundial with a horizontal gnomon.

Table 6 Complete constraint catalog for the conic sundial with horizontal gnomon.

Constraint ID	Physical Name	Formal Mathematical Expression
N1	Nodus on Polar Axis & Horizon	$\mathbf{N}_{\text{phys}} = (0, 0, 0)$; in math frame: $\mathbf{N} = d \hat{\mathbf{O}}$
G1	Gnomon in Meridian Plane	Horizontal rod at $X = 0, Z = 0$, extending northward ($+Y_w$)
C1	Apex on Polar Axis	$A_{\text{phys}} = d \hat{\mathbf{C}}$; in math frame: $A = (0, 0, 0)$
C2	Cone Axis = Polar Axis	$\hat{\mathbf{C}} = (0, \cos \varphi, \sin \varphi)$
C3	South-Opening Nappe	$\hat{\mathbf{O}} = -\hat{\mathbf{C}}$; $\mathbf{P} \cdot \hat{\mathbf{C}} < 0$
C4	Cavity Scale Bound	$d < H$ (nodus located within cone cavity)
C5	Rim Circularity	$R_{\text{rim}} = H \tan \alpha$ in plane orthogonal to $\hat{\mathbf{C}}$
E1	Equinox Invariant	$\ell_E = d$ for all α, φ
E2	Workshop Proportions	$x = \tan \alpha \tan \varepsilon$; $\ell_S = d/(1 - x)$, $\ell_W = d/(1 + x)$
E3	Noon Triplet Uncoupling	$\ell_W : \ell_E : \ell_S = (1 + x)^{-1} : 1 : (1 - x)^{-1}$ (independent of φ)
H1	Equinoctial Calibration	$H_E = 90^\circ$ at all latitudes (engraver's unit standard)
H2	Solstitial Hour Arc	$\cos H(\pm \varepsilon) = -\tan \varphi \tan(\pm \varepsilon)$
M1	Meridian Collinearity	$P_x = 0$ at $h = 0$ for all solar declinations δ

12 Conclusion

The construction geometry of the ancient conic sundial with a nodus on the polar axis is governed by an elegant uncoupling: the three seasonal noon marks on the meridian generator encode the cone half-angle α and scale d without any dependence on geographical latitude φ .

The closed-form Q -formula enables exact reconstruction of damaged fragments even when the apex is missing. Latitude φ must be derived from the solstitial hour division or the dressed reference planes of the block. By grounding computational 3D analysis in this rigorous proportional framework, archaeological sundials can be recovered not merely as geometric meshes, but as precise historical witnesses to ancient mathematical astronomy.

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